11.1. Gromov–Hausdorff distance.

(a) Show that if X, Y are two compact metric spaces with $d_{\mathrm{GH}}(X, Y) = 0$, then X and Y are isometric.

(b) Consider the two bounded, complete geodesic metric spaces X, Y defined as follows: X is a tree with one central vertex and edges of length $1-\frac{1}{2}, 1-\frac{1}{3}, 1-\frac{1}{4}, \dots$ attached to it, and Y is constructed similarly, but with an additional segment of length 1. Show that $d_{\mathrm{GH}}(X, Y) = 0$ despite X and Y being non-isometric.

11.2. Gromov–Hausdorff limits. Suppose that a sequence of metric spaces X_i converges in the Gromov–Hausdorff distance to a complete metric space X. Prove the first three assertions in Remark 6.19 (recall the definitions from Section 1.6). Hint for (1): Show that a complete metric space X is a length space if and only if for every pair of points $x, y \in X$ and $\epsilon > 0$ there exists an approximate midpoint $z \in X$ with $d(x, z), d(z, y) \leq \frac{1}{2}d(x, y) + \epsilon$.

11.3. Increase/collapse of dimension.

(a) Find a sequence of closed Riemannian surfaces that converges in the Gromov–Hausdorff distance to the unit cube $C := [0, 1]^3 \subset \mathbb{R}^3$ endowed with the l_1-distance $d_1(x, y) := \|x - y\|_1 = \sum_{n=1}^{3} |x_n - y_n|$.

(b) Find a sequence of Riemannian 3-spheres (S^3, g_k) that converges in the Gromov–Hausdorff distance to the 2-sphere $S^2(\frac{1}{2}) = \mathbb{M}^2_4$ of constant curvature 4.